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Multi-Cumulant and Non-Inferior Strategies for Multi-Player Pursuit-Evasion

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Abstract—The paper presents an extension of cost-cumulant control theory over a finite horizon for a class of two-team pursuit-evasion games wherein the evolution of the states of the game in response to decision strategies selected by pursuit and evasion teams from non-inferior sets of admissible controls is described by stochastic linear differential equation and integral quadratic cost. Since the sum of the aggregate cost functions of two teams is equal to zero, the amount that one team gains is equal to the amount of the other team loses. Both cooperation within each team and competition between the teams presumably exist. A direct dynamic programming approach for the Mayer optimization problem is used to solve for a multi-cumulant and non-inferior based solution when the members in each team measure the states and minimize the first k cumulants of the standard integral-quadratic cost associated with this special class of multi-player pursuit-evasion games.

I. INTRODUCTION

Since the 1950s, the work of Issacs [2] in deterministic pursuit-evasion game of a single pursuer and a single evader with perfect information and common knowledge has been greatly extended to pursuit-evasion with multiple pursuers and multiple evaders. Recent developments [6], [4] and references therein respectively treat probabilistic discrete-time as well deterministic continuous-time problems. To the best knowledge of the authors, there hasn't yet been any work done for multi-player pursuit-evasion differential game problems wherein the members in each team have common interests to statistically improve their payoffs at the expenses of the other members from the rival team. In particular, this paper is proposing a novel and innovative paradigm for non-inferior strategy selection using performance-measure statistics to provide not only a mechanism in which the common benefits of all members in each team can be optimized, but also an analytical tool which is used to characterize a complete statistical description of the global performance of the multi-player pursuit-evasion. The present work has extensive applications in multi-missile guidance and interception, military tactics, and strategic decision-making.

The paper is structured as follows. The necessary background in generating higher-order performance-measure statistics of the multi-player pursuit-evasion game is presented in Section II. These performance-measure statistics are then used to formulate the cost-cumulant control problem for the subject game. A precise mathematical formulation along with several problem statements of the multi-player

pursuit-evasion problem is summarized in Section III. Finally, a multi-cumulant, non-inferior and saddle-point solution and some remarks are presented in Sections IV and V.

II. PROBLEM FORMULATION

For analytical tractability, let's consider a special class of differential games whose dynamical systems of pursuers and evaders are linear and the cost functions are quadratic functions of the states and controls. For instance, a pursuit-evasion differential game with a team P with m_P pursuers, identified as m_1, \dots, m_P , and a team E with m_E evaders, identified as m_1, \dots, m_E , in an open subset of Hilbert space \mathcal{S} . Denote by $x_i^X(t) \triangleq x_i^X(t, \omega_i^X) : [t_0, t_f] \times \Omega_i^X \mapsto \mathbb{R}^{n_i^X}$ belonging to the Hilbert space $L_{\mathcal{F}_{it}^X}^2(\Omega_i^X; \mathcal{C}([t_0, t_f]; \mathbb{R}^{n_i^X}))$ of $\mathbb{R}^{n_i^X}$ -valued, square integrable processes on $[t_0, t_f]$ that are adapted to the σ -field \mathcal{F}_{it}^X generated by $w_i^X(t)$ with $E \left\{ \int_{t_0}^{t_f} (x_i^X)^T(\tau) x_i^X(\tau) d\tau \right\} < \infty$ the state variables for the members $i = 1, \dots, m_X$ in each team $X = P, E$ whose corresponding physical positions in S are described by

$$dx_i^X(t) = (A_i^X(t)x_i^X(t) + B_{ui}^X(t)u_i^X(t))dt + G_i^X(t)dw_i^X(t), \quad x_i^X(t_0) = x_{i0}^X \quad (1)$$

where the initial states x_{i0}^X are known. The input noises $w_i^X(t) \triangleq w_i^X(t, \omega_i^X) : [t_0, t_f] \times \Omega_i^X \mapsto \mathbb{R}^{p_i^X}$ are the p_i^X -dimensional stationary Wiener process defined with $\{\mathcal{F}_{it}^X\}_{t \geq 0}$ being its natural filtration on complete filtered probability spaces $(\Omega_i^X, \mathcal{F}_{it}^X, \{\mathcal{F}_{it}^X\}_{t \geq 0}, \mathcal{P}_i^X)$ over $[t_0, t_f]$ with the correlations of increments

$$E \{ [w_i^X(\tau) - w_i^X(\xi)][w_i^X(\tau) - w_i^X(\xi)]^T \} = W_i^X |\tau - \xi|,$$

and continuous-time coefficients $A_i^X \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n_i^X \times n_i^X})$, $B_{ui}^X \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n_i^X \times m_i^X})$, and $G_i^X \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n_i^X \times p_i^X})$. In (1), $u_i^X \in \mathcal{U}_i^X$ are the control vectors for the members in each team where $\mathcal{U}_i^X \in L_{\mathcal{F}_{it}^X}^2(\Omega_i^X; \mathcal{C}([t_0, t_f]; \mathbb{R}^{m_i^X}))$ are the sets of corresponding admissible control strategies in Hilbert space of $\mathbb{R}^{m_i^X}$ -valued, square integrable processes on $[t_0, t_f]$ that are adapted to the σ -field \mathcal{F}_{it}^X generated by $w_i^X(t)$. For simplicity of notation, let $x^X \triangleq [(x_1^X)^T, \dots, (x_{m_X}^X)^T]^T$, $u^X \triangleq [(u_1^X)^T, \dots, (u_{m_X}^X)^T]^T$, $A^X \triangleq \text{diag}(A_1^X, \dots, A_{m_X}^X)$, $B_u^X \triangleq \text{diag}(B_{u1}^X, \dots, B_{um_X}^X)$, and $G^X \triangleq \text{diag}(G_1^X, \dots, G_{m_X}^X)$. Then, the dynamic equations of multiple pursuers and evaders can be rewritten in a compact form as

$$dx^X(t) = (A^X(t)x^X(t) + B_u^X(t)u^X(t))dt + G^X(t)dw^X(t), \quad x^X(t_0) = x_0^X \quad (2)$$

and the aggregate dynamic equation of the multi-player pursuit-evasion differential game is then given by

$$dx(t) = (A(t)x(t) + B^P(t)u^P(t) + B^E(t)u^E(t))dt + G(t)dw(t), \quad x(t_0) = x_0 \quad (3)$$

where $A \triangleq \text{diag}(A^P, A^E)$, $B^P \triangleq [(B_u^P)^T, 0]^T$, $B^E \triangleq [0, (B_u^E)^T]^T$, $G \triangleq \text{diag}(G^P, G^E)$, $x \triangleq [(x^P)^T, (x^E)^T]^T$, $dw \triangleq [(dw^P)^T, (dw^E)^T]^T$, and $W \triangleq \text{diag}(W_1^P, \dots, W_{m_P}^P, W_1^E, \dots, W_{m_E}^E)$. Since not all evaders will be captured at the same time, the terminal time of the game, t_f should be defined based on the capture of all.

Definition 1: Terminal Time.

For any evaders $\{j\}_{j=1}^{m_E}$, assume that there exists a pursuer $\{i\}_{i=1}^{m_P}$ engaged with at least one evader. The capture time t_j of evader j is given by

$$t_j \triangleq \inf \{t \geq 0, \exists i: d(x_i^P(t), x_j^E(t)) \leq \epsilon, \epsilon \in \mathbb{R}^+\}. \quad (4)$$

Then, the terminal time t_f of the pursuit-evasion game is

$$t_f \triangleq \max_{1 \leq j \leq m_E} \{t_j\}. \quad (5)$$

Note that the terminal time could be infinity due to the inability of pursuers to capture some evaders whose physical and functional characteristics are superior to team P . Let $\mathcal{U}^P \triangleq \times_{i=1}^{m_P} \mathcal{U}_i^P$, $\mathcal{U}^E \triangleq \times_{j=1}^{m_E} \mathcal{U}_j^E$, $\mathcal{X} \triangleq \times_{i=1}^{m_P} \mathbb{R}^{n_i^P} \times \times_{j=1}^{m_E} \mathbb{R}^{n_j^E} \subset \mathcal{S}$. Then, associated with each $(u^P, u^E) \in \mathcal{U}^P \times \mathcal{U}^E$ is a finite-horizon integral quadratic form (IQF) cost $J_i^X: [t_0, t_f] \times \mathcal{X} \times \mathcal{U}^P \times \mathcal{U}^E \mapsto \mathbb{R}^+ \cup \{0\}$ for which the member i in team X , for $X = P, E$ attempts to optimize

$$\begin{aligned} J_i^X(t_0, x_0; u^P, u^E) &= x^T(t_f)Q_f^X x(t_f) \\ &+ \int_{t_0}^{t_f} \left[x^T(\tau)Q_i^X(\tau)x(\tau) + \sum_{j=1}^{m_P} (u_j^P)^T(\tau)R_{ij}^{XP}(\tau)u_j^P(\tau) \right. \\ &\quad \left. + \sum_{j=1}^{m_E} (u_j^E)^T(\tau)R_{ij}^{XE}(\tau)u_j^E(\tau) \right] d\tau, \end{aligned} \quad (6)$$

subject to the dynamics of the differential game (3) where $Q_{if}^X \in \mathbb{R}^{(\sum_{i=1}^{m_P} n_i^P + \sum_{j=1}^{m_E} n_j^E) \times (\sum_{i=1}^{m_P} n_i^P + \sum_{j=1}^{m_E} n_j^E)}$, $Q_i^X \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{(\sum_{i=1}^{m_P} n_i^P + \sum_{j=1}^{m_E} n_j^E) \times (\sum_{i=1}^{m_P} n_i^P + \sum_{j=1}^{m_E} n_j^E)})$, cross-coupling control inputs $R_{ij}^{XP} \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{m_j^P \times m_j^P})$ and $R_{ij}^{XE} \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{m_j^E \times m_j^E})$ are symmetric and positive semidefinite with $R_{ij}^{XP}(t)$ and $R_{ij}^{XE}(t)$ invertible.

Within a cooperative team X , it is of interest a negotiated solution among all members. A negotiation is done via mutual and enforceable agreements among team members. This solution is selected from the set of strategy m_X -tuples defined below.

Definition 2: Non-inferior Strategies.

The strategy m_X -tuple $v^X = (v_1^X, \dots, v_{m_X}^X)$ belongs to the noninferior set if, for any other strategy m_X -tuple $u^X: \{J_i^X(t_0, x_0; u^X, \cdot) \leq J_i^X(t_0, x_0; v^X, \cdot)\}$ only if $\{J_i^X(t_0, x_0; u^X, \cdot) = J_i^X(t_0, x_0; v^X, \cdot)\}$, for all $i = 1, \dots, m_X$.

Since the IQF costs (6) are convex functions on a convex set $\mathcal{U}^P \times \mathcal{U}^E$ with convex constraints (3), the problem of

solving for a set of non-inferior strategies within each team X with a vector cost criterion is equivalent to the problem of solving an $m_X - 1$ parameter family of optimal control problems with scalar cost criteria [5], [7]. Each non-inferior strategy m_X -tuple therefore minimizes the scalar criterion

$$J^X(t_0, x_0; u^P, u^E; \xi^X) = \sum_{i=1}^{m_X} \xi_i J_i^X(t_0, x_0; u^P, u^E), \quad (7)$$

where the set of team strategy profiles $\xi^X \in W^X$ is defined as follows

$$W^X \triangleq \left\{ \xi^X \in \mathbb{R}^{m_X} : \sum_{i=1}^{m_X} \xi_i^X = 1; 0 \leq \xi_i^X \leq 1 \right\}. \quad (8)$$

Let $Q_f^X \triangleq \sum_{i=1}^{m_X} \xi_i^X Q_{if}^X$, $Q^X \triangleq \sum_{i=1}^{m_X} \xi_i^X Q_i^X$, $R_j^{XP} \triangleq \sum_{i=1}^{m_X} \xi_i^X R_{ij}^{XP}$, and $R_j^{XE} \triangleq \sum_{i=1}^{m_X} \xi_i^X R_{ij}^{XE}$. Then the aggregate cost (7) can be written explicitly as follows

$$\begin{aligned} J^X(t_0, x_0; u^P, u^E; \xi^X) &= x^T(t_f)Q_f^X x(t_f) \\ &+ \int_{t_0}^{t_f} \left[x^T(\tau)Q^X(\tau)x(\tau) + \sum_{j=1}^{m_P} (u_j^P)^T(\tau)R_j^{XP}(\tau)u_j^P(\tau) \right. \\ &\quad \left. + \sum_{j=1}^{m_E} (u_j^E)^T(\tau)R_j^{XE}(\tau)u_j^E(\tau) \right] d\tau. \end{aligned} \quad (9)$$

For a compact notation, let $R^{XP} \triangleq \text{diag}(R_1^{XP}, \dots, R_{m_X}^{XP})$, and $R^{XE} \triangleq \text{diag}(R_1^{XE}, \dots, R_{m_X}^{XE})$. The negotiating cost (9) associated with team X then becomes

$$\begin{aligned} J^X(t_0, x_0; u^P, u^E; \xi^X) &= x^T(t_f)Q_f^X x(t_f) \\ &+ \int_{t_0}^{t_f} \left[x^T(\tau)Q^X(\tau)x(\tau) + (u^P)^T(\tau)R^{XP}(\tau)u^P(\tau) \right. \\ &\quad \left. + (u^E)^T(\tau)R^{XE}(\tau)u^E(\tau) \right] d\tau. \end{aligned} \quad (10)$$

In fact, the game is zero-sum only if $R^{PP} = -R^{EP} \triangleq R_P$, $R^{EE} = -R^{PE} \triangleq R_E$, $Q_f^P = -Q_f^E \triangleq Q_f$, and $Q^P = -Q^E \triangleq Q$. Substituting these results into (10), one obtains the zero-sum differential game cost

$$\begin{aligned} J(t_0, x_0; u^P, u^E) &= x^T(t_f)Q_f x(t_f) \\ &+ \int_{t_0}^{t_f} \left[x^T(\tau)Q(\tau)x(\tau) + (u^P)^T(\tau)R^P(\tau)u^P(\tau) \right. \\ &\quad \left. - (u^E)^T(\tau)R^E(\tau)u^E(\tau) \right] d\tau. \end{aligned} \quad (11)$$

In view of the linear system (3) and the quadratic performance-measure (11), it is reasonable to assume that both teams P and E choose their control actions from classes of linear memoryless-feedback strategies, $\gamma^P: [t_0, t_f] \times L_{\mathcal{F}_t}^2(\Omega; \mathcal{C}([t_0, t_f]; \mathcal{X})) \mapsto L_{\mathcal{F}_t}^2(\Omega; \mathcal{C}([t_0, t_f]; \mathcal{U}^P))$ and $\gamma^E: [t_0, t_f] \times L_{\mathcal{F}_t}^2(\Omega; \mathcal{C}([t_0, t_f]; \mathcal{X})) \mapsto L_{\mathcal{F}_t}^2(\Omega; \mathcal{C}([t_0, t_f]; \mathcal{U}^E))$

$$u^P(t) = \gamma^P(t, x(t)) = K^P(t)x(t), \quad (12)$$

$$u^E(t) = \gamma^E(t, x(t)) = K^E(t)x(t), \quad (13)$$

where $K^P \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{\sum_{i=1}^{m_P} m_i^P \times (\sum_{i=1}^{m_P} n_i^P + \sum_{j=1}^{m_E} n_j^E)})$ and $K^E \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{\sum_{j=1}^{m_E} m_j^E \times (\sum_{i=1}^{m_P} n_i^P + \sum_{j=1}^{m_E} n_j^E)})$ are admissible gains for teams P and E . For the given initial condition

$(t_0, x_0) \in [t_0, t_f] \times \mathcal{S}$ and control strategies subject to (12)-(13), the dynamics of the game (3) is then given by

$$dx(t) = [A(t) + B^P(t)K^P(t) + B^E(t)K^E(t)]x(t)dt + G(t)dw(t), \quad x(t_0) = x_0, \quad (14)$$

and its IQF cost also follows

$$J(t_0, x_0; K^P, K^E) = x^T(t_f)Q_f x(t_f) + \int_{t_0}^{t_f} x^T(\tau)[Q(\tau) + K^{PT}(\tau)R^P(\tau)K^P(\tau) - K^{ET}(\tau)R^E(\tau)K^E(\tau)]x(\tau)d\tau. \quad (15)$$

It is now necessary to develop a procedure for generating cost cumulants for the zero-sum stochastic differential game by adapting the parametric method in [3] to characterize a moment-generating function. These cost cumulants are then used to form a performance index in the cost-cumulant control optimization. This approach begins with a replacement of the initial condition (t_0, x_0) by any arbitrary pair (α, x_α) . Thus, for the given admissible feedback gains K^P and K^E , the cost functional (15) is seen as the “cost-to-go”, $J(\alpha, x_\alpha)$. The moment-generating function of the vector-valued random process (14) is given by

$$\varphi(\alpha, x_\alpha; \theta) \triangleq E\{\exp(\theta J(\alpha, x_\alpha))\}, \quad (16)$$

where the scalar $\theta \in \mathbb{R}^+$ is a small parameter. Thus, the cumulant-generating function immediately follows

$$\psi(\alpha, x_\alpha; \theta) \triangleq \ln\{\varphi(\alpha, x_\alpha; \theta)\}, \quad (17)$$

in which $\ln\{\cdot\}$ denotes the natural logarithmic transformation of an enclosed entity.

Theorem 1: Cost Cumulant-Generating Function.

For all $\alpha \in [t_0, t_f]$ and the small parameter $\theta \in \mathbb{R}^+$, define

$$\varphi(\alpha, x_\alpha; \theta) \triangleq \varrho(\alpha, \theta) \exp(x_\alpha^T \Upsilon(\alpha, \theta) x_\alpha), \quad (18)$$

$$v(\alpha, \theta) \triangleq \ln\{\varrho(\alpha, \theta)\}. \quad (19)$$

Then the cost cumulant-generating function is expressed as

$$\psi(\alpha, x_\alpha; \theta) = x_\alpha^T \Upsilon(\alpha, \theta) x_\alpha + v(\alpha, \theta), \quad (20)$$

in which the scalar solution $v(\alpha, \theta)$ solves the backward-in-time differential equation with $v(t_f, \theta) = 0$

$$\frac{d}{d\alpha} v(\alpha, \theta) = -\text{Tr}\{\Upsilon(\alpha, \theta)G(\alpha)WG^T(\alpha)\}, \quad (21)$$

whereas $\Upsilon(\alpha, \theta)$ satisfies the backward-in-time differential equation together with $\Upsilon(t_f, \theta) = \theta Q_f$

$$\begin{aligned} \frac{d}{d\alpha} \Upsilon(\alpha, \theta) = & -[A(\alpha) + B^P(\alpha)K^P(\alpha) + B^E(\alpha)K^E(\alpha)]^T \Upsilon(\alpha, \theta) \\ & - \Upsilon(\alpha, \theta)[A(\alpha) + B^P(\alpha)K^P(\alpha) + B^E(\alpha)K^E(\alpha)] \\ & - 2\Upsilon(\alpha, \theta)G(\alpha)WG^T(\alpha)\Upsilon(\alpha, \theta) - \theta[Q(\alpha) \\ & + K^{PT}(\alpha)R^P(\alpha)K^P(\alpha) - K^{ET}(\alpha)R^E(\alpha)K^E(\alpha)]. \end{aligned} \quad (22)$$

Meanwhile, $\varrho(\alpha, \theta)$ satisfies the backward-in-time differential equation with $\varrho(t_f, \theta) = 1$

$$\frac{d}{d\alpha} \varrho(\alpha, \theta) = -\varrho(\alpha, \theta) \text{Tr}\{\Upsilon(\alpha, \theta)G(\alpha)WG^T(\alpha)\}. \quad (23)$$

Proof. For any θ given, let $\varpi(\alpha, x_\alpha; \theta) \triangleq \exp(\theta J(\alpha, x_\alpha))$ then the moment-generating function becomes

$$\varphi(\alpha, x_\alpha; \theta) = E\{\varpi(\alpha, x_\alpha; \theta)\},$$

with the time derivative of

$$\begin{aligned} \frac{d}{d\alpha} \varphi(\alpha, x_\alpha; \theta) = & -\varphi(\alpha, x_\alpha; \theta) \theta x_\alpha^T [Q(\alpha) \\ & + K^{PT}(\alpha)R^P(\alpha)K^P(\alpha) - K^{ET}(\alpha)R^E(\alpha)K^E(\alpha)] x_\alpha. \end{aligned}$$

Using the standard Ito's formula, one get

$$\begin{aligned} d\varphi(\alpha, x_\alpha; \theta) = & E\{d\varpi(\alpha, x_\alpha; \theta)\}, \\ = & E\{\varpi_\alpha(\alpha, x_\alpha; \theta) d\alpha + \varpi_{x_\alpha}(\alpha, x_\alpha; \theta) dx_\alpha \\ & + \frac{1}{2} \text{Tr}\{\varpi_{x_\alpha x_\alpha}(\alpha, x_\alpha; \theta)G(\alpha)WG^T(\alpha)\} d\alpha\}, \\ = & \varphi_\alpha(\alpha, x_\alpha; \theta) d\alpha \\ & + \varphi_{x_\alpha}(\alpha, x_\alpha; \theta)[A(\alpha) + B^P(\alpha)K^P(\alpha) + B^E(\alpha)K^E(\alpha)]x_\alpha d\alpha \\ & + \frac{1}{2} \text{Tr}\{\varphi_{x_\alpha x_\alpha}(\alpha, x_\alpha; \theta)G(\alpha)WG^T(\alpha)\} d\alpha, \end{aligned}$$

which with the definition (18) leads to

$$\begin{aligned} & -\varphi(\alpha, x_\alpha; \theta) \theta x_\alpha^T [Q(\alpha) + K^{PT}(\alpha)R^P(\alpha)K^P(\alpha) \\ & - K^{ET}(\alpha)R^E(\alpha)K^E(\alpha)] x_\alpha = \frac{d}{d\alpha} \varrho(\alpha, \theta) \varphi(\alpha, x_\alpha; \theta) \\ & + \varphi(\alpha, x_\alpha; \theta) x_\alpha^T \frac{d}{d\alpha} \Upsilon(\alpha, \theta) x_\alpha + \varphi(\alpha, x_\alpha; \theta) \left\{ x_\alpha^T [A(\alpha) \right. \\ & \quad + B^P(\alpha)K^P(\alpha) + B^E(\alpha)K^E(\alpha)]^T \Upsilon(\alpha, \theta) x_\alpha \\ & \quad + x_\alpha^T \Upsilon_\alpha(\alpha, \theta) [A(\alpha) + B^P(\alpha)K^P(\alpha) + B^E(\alpha)K^E(\alpha)] x_\alpha \Big\} \\ & \quad + \varphi(\alpha, x_\alpha; \theta) \left\{ 2x_\alpha^T \Upsilon(\alpha, \theta)G(\alpha)WG^T(\alpha)\Upsilon(\alpha, \theta) x_\alpha \right. \\ & \quad \left. + \text{Tr}\{\Upsilon(\alpha, \theta)G(\alpha)WG^T(\alpha)\} \right\}. \end{aligned}$$

To have constant and quadratic terms being independent of x_α , it requires that

$$\begin{aligned} \frac{d}{d\alpha} \Upsilon(\alpha, \theta) = & -[A(\alpha) + B^P(\alpha)K^P(\alpha) + B^E(\alpha)K^E(\alpha)]^T \Upsilon(\alpha, \theta) \\ & - \Upsilon(\alpha, \theta)[A(\alpha) + B^P(\alpha)K^P(\alpha) + B^E(\alpha)K^E(\alpha)] \\ & - 2\Upsilon(\alpha, \theta)G(\alpha)WG^T(\alpha)\Upsilon(\alpha, \theta) \\ & - \theta[Q(\alpha) + K^{PT}(\alpha)R^P(\alpha)K^P(\alpha) - K^{ET}(\alpha)R^E(\alpha)K^E(\alpha)], \\ \frac{d}{d\alpha} \varrho(\alpha, \theta) = & -\varrho(\alpha, \theta) \text{Tr}\{\Upsilon(\alpha, \theta)G(\alpha)WG^T(\alpha)\}, \end{aligned}$$

with the terminal conditions $\Upsilon(t_f, \theta) = \theta Q_f$ and $\varrho(t_f, \theta) = 1$. Finally, the remaining backward-in-time differential equation satisfied by $v(\alpha, \theta)$ is given by

$$\frac{d}{d\alpha} v(\alpha, \theta) = -\text{Tr} \{ \Upsilon(\alpha, \theta) G(\alpha) W G^T(\alpha) \}, \quad v(t_f, \theta) = 0$$

which completes the proof.

The MacLaurin expansion of the cumulant-generating function is used to generate cost cumulants for the multi-player pursuit-evasion game

$$\psi(\alpha, x_\alpha; \theta) = \sum_{i=1}^{\infty} \kappa_i(\alpha, x_\alpha) \frac{\theta^i}{i!} = \sum_{i=1}^{\infty} \frac{\partial^{(i)}}{\partial \theta^{(i)}} \psi(\alpha, x_\alpha; \theta) \Big|_{\theta=0} \frac{\theta^i}{i!} \quad (24)$$

in which $\kappa_i(\alpha, x_\alpha)$'s are the cost cumulants. Note that the series coefficients can be computed by using (20)

$$\begin{aligned} \frac{\partial^{(i)}}{\partial \theta^{(i)}} \psi(\alpha, x_\alpha; \theta) \Big|_{\theta=0} &= x_\alpha^T \frac{\partial^{(i)}}{\partial \theta^{(i)}} \Upsilon(\alpha, \theta) \Big|_{\theta=0} x_\alpha \\ &+ \frac{\partial^{(i)}}{\partial \theta^{(i)}} v(\alpha, \theta) \Big|_{\theta=0}. \end{aligned} \quad (25)$$

Cost cumulants for the stochastic differential game problem can be obtained using (24) and (25) as follows

$$\kappa_i(\alpha, x_\alpha) = x_\alpha^T \frac{\partial^{(i)}}{\partial \theta^{(i)}} \Upsilon(\alpha, \theta) \Big|_{\theta=0} x_\alpha + \frac{\partial^{(i)}}{\partial \theta^{(i)}} v(\alpha, \theta) \Big|_{\theta=0}, \quad (26)$$

for any finite $1 \leq i < \infty$. For notational convenience, the following definitions are needed in place

$$H(\alpha, i) \triangleq \frac{\partial^{(i)}}{\partial \theta^{(i)}} \Upsilon(\alpha, \theta) \Big|_{\theta=0}; \quad D(\alpha, i) \triangleq \frac{\partial^{(i)}}{\partial \theta^{(i)}} v(\alpha, \theta) \Big|_{\theta=0}. \quad (27)$$

Theorem 2: Cumulants in Multi-Player Pursuit-Evasion.

Suppose the multi-player pursuit-evasion game is characterized by (14)-(15) where (A, B^P) and (A, B^E) are uniformly stabilizable. Two teams presumably choose their control strategies $(u^P(t), u^E(t)) = (K^P(t)x(t), K^E(t)x(t))$. For given $k \in \mathbb{Z}^+$, $\xi^P \in W^P$, and $\xi^E \in W^E$, the k^{th} cost cumulant in multi-player pursuit-evasion is computed by

$$\kappa_k(t_0, x_0; \xi^P, \xi^E; K^P, K^E) = x_0^T H(t_0, k) x_0 + D(t_0, k) \quad (28)$$

in which the cumulant-building variables $\{H(\alpha, i)\}_{i=1}^k$ and $\{D(\alpha, i)\}_{i=1}^k$ evaluated at $\alpha = t_0$ satisfy the following differential equations (with the dependence of $H(\alpha, i)$ and $D(\alpha, i)$ upon the admissible gains K^P and K^E suppressed)

$$\begin{aligned} \frac{d}{d\alpha} H(\alpha, 1) &= \\ &- [A(\alpha) + B^P(\alpha)K^P(\alpha) + B^E(\alpha)K^E(\alpha)]^T H(\alpha, 1) \\ &- H(\alpha, 1) [A(\alpha) + B^P(\alpha)K^P(\alpha) + B^E(\alpha)K^E(\alpha)] - Q(\alpha) \\ &- K^{PT}(\alpha)R^P(\alpha)K^P(\alpha) + K^{ET}(\alpha)R^E(\alpha)K^E(\alpha), \end{aligned} \quad (29)$$

and, for $2 \leq i \leq k$

$$\begin{aligned} \frac{d}{d\alpha} H(\alpha, i) &= \\ &- [A(\alpha) + B^P(\alpha)K^P(\alpha) + B^E(\alpha)K^E(\alpha)]^T H(\alpha, i) \\ &- H(\alpha, i) [A(\alpha) + B^P(\alpha)K^P(\alpha) + B^E(\alpha)K^E(\alpha)] \\ &- \sum_{j=1}^{i-1} \frac{2i!}{j!(i-j)!} H(\alpha, j) G(\alpha) W G^T(\alpha) H(\alpha, i-j), \end{aligned} \quad (30)$$

together with $1 \leq i \leq k$

$$\frac{d}{d\alpha} D(\alpha, i) = -\text{Tr} \{ H(\alpha, i) G(\alpha) W G^T(\alpha) \}, \quad (31)$$

where the terminal conditions $H(t_f, 1) = Q_f$, $H(t_f, i) = 0$ for $2 \leq i \leq k$ and $D(t_f, i) = 0$ for $1 \leq i \leq k$.

Proof. The cost cumulant expression in (28) is readily justified by using the result (26) and the definitions (27). What remains is to show that the solutions $H(\alpha, i)$ and $D(\alpha, i)$ for $1 \leq i \leq k$ indeed satisfy the equations (29)-(31). Note that the equations (29)-(31) satisfied by the solutions $H(\alpha, i)$ and $D(\alpha, i)$ can be obtained by repeatedly taking the derivative with respect to θ of the equations (21)-(22) together with the assumption $A(\alpha) + B^P(\alpha)K^P(\alpha) + B^E(\alpha)K^E(\alpha)$, stable for all $\alpha \in [t_0, t_f]$.

III. PROBLEM STATEMENTS

In the subsequent development, the subset of symmetric matrices of the vector space of all $n \times n$ matrices with real elements is denoted by \mathbb{S}^n where $n \triangleq \sum_{i=1}^{m_P} n_i^P + \sum_{j=1}^{m_E} n_j^E$. Now let k -tuple variables \mathcal{H} and \mathcal{D} be defined as follows $\mathcal{H}(\cdot) \triangleq (\mathcal{H}_1(\cdot), \dots, \mathcal{H}_k(\cdot))$ and $\mathcal{D}(\cdot) \triangleq (\mathcal{D}_1(\cdot), \dots, \mathcal{D}_k(\cdot))$ for each element $\mathcal{H}_i \in \mathcal{C}^1([t_0, t_f]; \mathbb{S}^n)$ of \mathcal{H} and $\mathcal{D}_i \in \mathcal{C}^1([t_0, t_f]; \mathbb{R})$ of \mathcal{D} having the representations $\mathcal{H}_i(\cdot) = H(\cdot, i)$ and $\mathcal{D}_i(\cdot) = D(\cdot, i)$ with the right members satisfying the dynamic equations (29)-(31) on the horizon $[t_0, t_f]$. For notational tractability, the following mappings are introduced

$$\begin{aligned} \mathcal{F}_i &: [t_0, t_f] \times (\mathbb{S}^n)^k \times \mathbb{R}^{m^P \times n} \times \mathbb{R}^{m^E \times n} \mapsto \mathbb{S}^n \\ \mathcal{G}_i &: [t_0, t_f] \times (\mathbb{S}^n)^k \mapsto \mathbb{R} \end{aligned}$$

where $m^P \triangleq \sum_{i=1}^{m_P} m_i^P$, $m^E \triangleq \sum_{j=1}^{m_E} m_j^E$, and the actions are given by

$$\begin{aligned} \mathcal{F}_1(\alpha, \mathcal{H}, K^P, K^E) &\triangleq \\ &- [A(\alpha) + B^P(\alpha)K^P(\alpha) + B^E(\alpha)K^E(\alpha)]^T \mathcal{H}_1(\alpha) \\ &- \mathcal{H}_1(\alpha) [A(\alpha) + B^P(\alpha)K^P(\alpha) + B^E(\alpha)K^E(\alpha)] \\ &- Q(\alpha) - K^{PT}(\alpha)R^P(\alpha)K^P(\alpha) + K^{ET}(\alpha)R^E(\alpha)K^E(\alpha), \\ \mathcal{F}_i(\alpha, \mathcal{H}, K^P, K^E) &\triangleq \\ &- [A(\alpha) + B^P(\alpha)K^P(\alpha) + B^E(\alpha)K^E(\alpha)]^T \mathcal{H}_i(\alpha) \\ &- \mathcal{H}_i(\alpha) [A(\alpha) + B^P(\alpha)K^P(\alpha) + B^E(\alpha)K^E(\alpha)] \\ &- \sum_{j=1}^{i-1} \frac{2i!}{j!(i-j)!} \mathcal{H}_j(\alpha) G(\alpha) W G^T(\alpha) \mathcal{H}_{i-j}(\alpha), \quad 2 \leq i \leq k, \\ \mathcal{G}_i(\alpha, \mathcal{H}) &\triangleq -\text{Tr} \{ \mathcal{H}_i(\alpha) G(\alpha) W G^T(\alpha) \}, \quad 1 \leq i \leq k. \end{aligned}$$

For a compact formulation, the following product mappings are introduced

$$\mathcal{F}_1 \times \cdots \times \mathcal{F}_k : [t_0, t_f] \times (\mathbb{S}^n)^k \times \mathbb{R}^{m^P \times n} \times \mathbb{R}^{m^E \times n} \mapsto (\mathbb{S}^n)^k$$

$$\mathcal{G}_1 \times \cdots \times \mathcal{G}_k : [t_0, t_f] \times (\mathbb{S}^n)^k \mapsto \mathbb{R}^k$$

along with the corresponding notations $\mathcal{F} \triangleq \mathcal{F}_1 \times \cdots \times \mathcal{F}_k$ and $\mathcal{G} \triangleq \mathcal{G}_1 \times \cdots \times \mathcal{G}_k$. Thus, the dynamic equations of motion (29)-(31) can be rewritten as follows

$$\frac{d}{d\alpha} \mathcal{H}(\alpha) = \mathcal{F}(\alpha, \mathcal{H}(\alpha), K^P(\alpha), K^E(\alpha)), \quad \mathcal{H}(t_f) \quad (32)$$

$$\frac{d}{d\alpha} \mathcal{D}(\alpha) = \mathcal{G}(\alpha, \mathcal{H}(\alpha)), \quad \mathcal{D}(t_f) \quad (33)$$

where the terminal values $\mathcal{H}(t_f) = (Q_f, 0, \dots, 0)$ and $\mathcal{D}(t_f) = (0, \dots, 0)$.

Note that the product system uniquely determines \mathcal{H} and \mathcal{D} once the admissible feedback gains K^P and K^E are specified. Hence, \mathcal{H} and \mathcal{D} are considered as $\mathcal{H}(\cdot, K^P, K^E)$ and $\mathcal{D}(\cdot, K^P, K^E)$, respectively. The performance index in cost-cumulant control can now be formulated in the admissible feedback gains K^P and K^E .

Definition 3: Performance Index.

Fix $k \in \mathbb{Z}^+$ and $\mu = \{\mu_i \geq 0\}_{i=1}^k$ with $\mu_1 > 0$. Then for given (t_0, x_0) , $\xi^P \in W^P$, and $\xi^E \in W^E$, the performance index $\phi_0 : [t_0, t_f] \times (\mathbb{S}^n)^k \times \mathbb{R}^k \mapsto \mathbb{R}^+$ of the cost-cumulant control is defined as

$$\begin{aligned} & \phi_0(t_0, \mathcal{H}(t_0, K^P, K^E), \mathcal{D}(t_0, K^P, K^E)) \\ & \triangleq \sum_{i=1}^k \mu_i \kappa_i(K^P, K^E) \\ & = \sum_{i=1}^k \mu_i [x_0^T \mathcal{H}_i(t_0, K^P, K^E) x_0 + \mathcal{D}_i(t_0, K^P, K^E)] \end{aligned} \quad (34)$$

where the parametric design freedom μ_i mutually chosen by two non-cooperative teams represent different levels of influence as they deem important to the overall cost distribution of the multi-player pursuit-evasion game and solutions $\{\mathcal{H}_i(t_0, K^P, K^E) \geq 0\}_{i=1}^k$ and $\{\mathcal{D}_i(t_0, K^P, K^E) \geq 0\}_{i=1}^k$ evaluated at $\alpha = t_0$ satisfy the equations (32)-(33).

For the given terminal data $(t_f, \mathcal{H}_f, \mathcal{D}_f)$, the classes $\mathcal{K}_{t_f, \mathcal{H}_f, \mathcal{D}_f; \xi^P, \xi^E; \mu}^P$ and $\mathcal{K}_{t_f, \mathcal{H}_f, \mathcal{D}_f; \xi^P, \xi^E; \mu}^E$ of admissible feedback gains may be defined as follows.

Definition 4: Admissible Feedback Gain Strategies.

Let the compact subsets $\overline{K}^P \subset \mathbb{R}^{m^P \times n}$ and $\overline{K}^E \subset \mathbb{R}^{m^E \times n}$ be the sets of allowable gain values. For given $\xi^P \in W^P$, $\xi^E \in W^E$, $k \in \mathbb{Z}^+$, and $\mu = \{\mu_i \geq 0\}_{i=1}^k$ with $\mu_1 > 0$, the sets of admissible control strategies $\mathcal{K}_{t_f, \mathcal{H}_f, \mathcal{D}_f; \xi^P, \xi^E; \mu}^P$ and $\mathcal{K}_{t_f, \mathcal{H}_f, \mathcal{D}_f; \xi^P, \xi^E; \mu}^E$ are assumed to be the classes of $\mathcal{C}([t_0, t_f]; \mathbb{R}^{m^P \times n})$ and $\mathcal{C}([t_0, t_f]; \mathbb{R}^{m^E \times n})$ with values $K^P(\cdot) \in \overline{K}^P$ and $K^E(\cdot) \in \overline{K}^E$ for which solutions to the dynamic equations of motion (32)-(33) exist on the finite horizon $[t_0, t_f]$.

Then one may state the optimization problem for the zero-sum stochastic differential game.

Definition 5: Optimization Problem.

Fix $\xi^P \in W^P$, $\xi^E \in W^E$, $k \in \mathbb{Z}^+$, and $\mu = \{\mu_i \geq 0\}_{i=1}^k$

with $\mu_1 > 0$. Then the optimization problem for multi-player pursuit-evasion over $[t_0, t_f]$ is given by

$$\begin{aligned} & \min_{K^P(\cdot) \in \mathcal{K}_{t_f, \mathcal{H}_f, \mathcal{D}_f; \xi^P, \xi^E; \mu}^P} \max_{K^E(\cdot) \in \mathcal{K}_{t_f, \mathcal{H}_f, \mathcal{D}_f; \xi^P, \xi^E; \mu}^E} \\ & \phi_0(t_0, \mathcal{H}(t_0, K^P, K^E), \mathcal{D}(t_0, K^P, K^E)) \end{aligned} \quad (35)$$

subject to the dynamic equations (32)-(33) for $\alpha \in [t_0, t_f]$.

It is worth mentioning that the subject optimization is an initial cost problem, in contrast with the more traditional terminal cost class of investigations. One may address an initial cost problem by introducing changes of variables which convert it to a terminal cost problem. However, this modifies the natural context of cost cumulants, which it is preferable to retain. Instead, one may take a more direct dynamic programming approach to the initial cost problem. Such an approach is illustrative of the more general concept of the principle of optimality, an idea tracing its roots back to the 17th century.

As a tenet of transition from the principle of optimality, a family of games based on different starting points is now of concerned. Let's begin by considering an interlude of time, ε in mid-play. At its commencement the path has reached some definitive point. Consider all possible $(\mathcal{H}, \mathcal{D})$ which may be reached at the end of the interlude for all possible choices of (K^P, K^E) . Suppose that for each endpoint, the game beginning there has already been solved. Then the value function $\mathcal{V}(\varepsilon, \mathcal{H}, \mathcal{D})$ resulting from each choice of (K^P, K^E) is known, and they are to be so chosen as to render it minimax. As the duration of the interlude approaches t_f , this leads to a sufficient condition to Hamilton-Jacobi-Isaacs (HJI) equation.

Definition 6: Playable Set.

Let the playable set \mathcal{Q} be defined as follows

$$\begin{aligned} \mathcal{Q} \triangleq & \left\{ (\varepsilon, \mathcal{Y}, \mathcal{Z}) \in [t_0, t_f] \times (\mathbb{S}^n)^k \times \mathbb{R}^k \text{ such that} \right. \\ & \left. \mathcal{K}_{\varepsilon, \mathcal{Y}, \mathcal{Z}; \xi^P, \xi^E; \mu}^P \times \mathcal{K}_{\varepsilon, \mathcal{Y}, \mathcal{Z}; \xi^P, \xi^E; \mu}^E \neq \emptyset \right\}. \end{aligned}$$

The fundamental theorem of calculus and stochastic differential rules can be used to derive a saddle point.

Theorem 3: Existence of a Saddle Point.

Fix $k \in \mathbb{Z}^+$ and $\mu = \{\mu_i \geq 0\}_{i=1}^k$ with $\mu_1 > 0$. Then for given (t_0, x_0) , $\xi^P \in W^P$ and $\xi^E \in W^E$, there exists a saddle point $(K^{P*}, K^{E*}) \in \mathcal{K}_{t_f, \mathcal{H}_f, \mathcal{D}_f; \xi^P, \xi^E; \mu}^P \times \mathcal{K}_{t_f, \mathcal{H}_f, \mathcal{D}_f; \xi^P, \xi^E; \mu}^E$ such that there holds

$$\begin{aligned} & \phi_0(t_0, \mathcal{H}(t_0, K^{P*}, K^{E*}), \mathcal{D}(t_0, K^{P*}, K^{E*})) \\ & \leq \phi_0(t_0, \mathcal{H}(t_0, K^{P*}, K^{E*}), \mathcal{D}(t_0, K^{P*}, K^{E*})) \\ & \leq \phi_0(t_0, \mathcal{H}(t_0, K^P, K^{E*}), \mathcal{D}(t_0, K^P, K^{E*})) \end{aligned}$$

Therefore, the existence of a saddle point yields both necessary and sufficient conditions for the minimax problem to be equivalent to the corresponding maximin problem.

Theorem 4: Differentiability of Value Function.

Let admissible feedback gains $K^{P*}(\alpha, \mathcal{H}, \mathcal{D})$ and $K^{E*}(\alpha, \mathcal{H}, \mathcal{D})$ constitute a saddle point. Further, let $t_0(\varepsilon, \mathcal{Y}, \mathcal{Z})$ and $(\mathcal{H}(t_0(\varepsilon, \mathcal{Y}, \mathcal{Z}); \varepsilon, \mathcal{Y}), \mathcal{D}(t_0(\varepsilon, \mathcal{Y}, \mathcal{Z}); \varepsilon, \mathcal{Z}))$

be the initial time and initial states for the trajectories of

$$\begin{aligned}\frac{d}{d\alpha}\mathcal{H}(\alpha) &= \mathcal{F}(\alpha, \mathcal{H}, K^{P*}(\alpha, \mathcal{H}, \mathcal{D}), K^{E*}(\alpha, \mathcal{H}, \mathcal{D})), \\ \frac{d}{d\alpha}\mathcal{D}(\alpha) &= \mathcal{G}(\alpha, \mathcal{H}),\end{aligned}$$

with the terminal condition $(\varepsilon, \mathcal{Y}, \mathcal{Z})$. Then, the value function $\mathcal{V}(\varepsilon, \mathcal{Y}, \mathcal{Z})$ is differentiable at each point at which $t_0(\varepsilon, \mathcal{Y}, \mathcal{Z})$ and $\mathcal{H}(t_0(\varepsilon, \mathcal{Y}, \mathcal{Z}); \varepsilon, \mathcal{Y})$ and $\mathcal{D}(t_0(\varepsilon, \mathcal{Y}, \mathcal{Z}); \varepsilon, \mathcal{Z})$ are differentiable with respect to $(\varepsilon, \mathcal{Y}, \mathcal{Z})$.

Moreover, if the value function is continuously differentiable then such a saddle point is unique.

Theorem 5: HJI Equation-Mayer Problem.

Let $(\varepsilon, \mathcal{Y}, \mathcal{Z})$ be any interior point of the playable set \mathcal{Q} at which the value function $\mathcal{V}(\varepsilon, \mathcal{Y}, \mathcal{Z})$ is differentiable. If there exist a saddle point $(K^{P*}, K^{E*}) \in \mathcal{K}_{\varepsilon, \mathcal{Y}, \mathcal{Z}; \xi^P, \xi^E; \mu}^P \times \mathcal{K}_{\varepsilon, \mathcal{Y}, \mathcal{Z}; \xi^P, \xi^E; \mu}^E$, then the partial differential equation of the pursuit-evasion differential games

$$\begin{aligned}0 = & \min_{K^P \in \bar{K}^P} \max_{K^E \in \bar{K}^E} \left\{ \frac{\partial}{\partial \varepsilon} \mathcal{V}(\varepsilon, \mathcal{Y}, \mathcal{Z}) \right. \\ & + \frac{\partial}{\partial \text{vec}(\mathcal{Y})} \mathcal{V}(\varepsilon, \mathcal{Y}, \mathcal{Z}) \cdot \text{vec}(\mathcal{F}(\varepsilon, \mathcal{Y}, K^P, K^E)) \\ & \left. + \frac{\partial}{\partial \text{vec}(\mathcal{Z})} \mathcal{V}(\varepsilon, \mathcal{Y}, \mathcal{Z}) \cdot \text{vec}(\mathcal{G}(\varepsilon, \mathcal{Y})) \right\} \quad (36)\end{aligned}$$

is satisfied together with

$$\mathcal{V}(t_0, \mathcal{H}_0, \mathcal{D}_0) = \phi_0(t_0, \mathcal{H}_0, \mathcal{D}_0)$$

and $\text{vec}(\cdot)$ the vectorizing operator of enclosed entities.

IV. SADDLE-POINT STRATEGIES

The approach of obtaining a saddle-point solution requires parametrization of the terminal time and states of the optimization problem as $(\varepsilon, \mathcal{Y}, \mathcal{Z})$ rather than $(t_f, \mathcal{H}_f, \mathcal{D}_f)$. That is, for $\varepsilon \in [t_0, t_f]$ and $1 \leq i \leq k$, the states of the system (32)-(33) defined on the interval $[t_0, \varepsilon]$ have the terminal values denoted by $\mathcal{H}(\varepsilon) = \mathcal{Y}$ and $\mathcal{D}(\varepsilon) = \mathcal{Z}$. Observe that the cumulant-based performance index (34) is quadratic affine in terms of arbitrarily fixed x_0 . This suggests a solution to the HJI equation (36) may be sought in the form

$$\begin{aligned}\mathcal{W}(\varepsilon, \mathcal{Y}, \mathcal{Z}) \\ = x_0^T \sum_{i=1}^k \mu_i (\mathcal{Y}_i + \mathcal{E}_i(\varepsilon)) x_0 + \sum_{i=1}^k \mu_i (\mathcal{Z}_i + \mathcal{T}_i(\varepsilon)), \quad (37)\end{aligned}$$

where these parametric functions of time $\mathcal{E}_i \in \mathcal{C}^1([t_0, t_f]; \mathbb{S}^n)$ and $\mathcal{T}_i \in \mathcal{C}^1([t_0, t_f]; \mathbb{R})$ are to be determined.

Theorem 6: Time Derivative of a Candidate Function.

Fix $k \in \mathbb{Z}^+$ and let $(\varepsilon, \mathcal{Y}, \mathcal{Z})$ be any interior point of the reachable set \mathcal{Q} at which the real-valued function (37) is differentiable. Then, the time derivative of $\mathcal{W}(\varepsilon, \mathcal{Y}, \mathcal{Z})$ is

found to be

$$\begin{aligned}\frac{d}{d\varepsilon} \mathcal{W}(\varepsilon, \mathcal{Y}, \mathcal{Z}) &= \sum_{i=1}^k \mu_i \left(\mathcal{G}_i(\varepsilon, \mathcal{Y}) + \frac{d}{d\varepsilon} \mathcal{T}_i(\varepsilon) \right) \\ &+ x_0^T \sum_{i=1}^k \mu_i \left(\mathcal{F}_i(\varepsilon, \mathcal{Y}, K^P, K^E) + \frac{d}{d\varepsilon} \mathcal{E}_i(\varepsilon) \right) x_0. \quad (38)\end{aligned}$$

The substitution of this hypothesized solution (37) into the HJI equation (36) and making use of the result (38) yield

$$\begin{aligned}0 = & \min_{K^P \in \bar{K}^P} \max_{K^E \in \bar{K}^E} \left\{ \frac{\partial}{\partial \varepsilon} \mathcal{W}(\varepsilon, \mathcal{Y}, \mathcal{Z}) \right. \\ & + \frac{\partial}{\partial \text{vec}(\mathcal{Y})} \mathcal{W}(\varepsilon, \mathcal{Y}, \mathcal{Z}) \cdot \text{vec}(\mathcal{F}_i(\varepsilon, \mathcal{Y}, K^P, K^E)) \\ & \left. + \frac{\partial}{\partial \text{vec}(\mathcal{Z})} \mathcal{W}(\varepsilon, \mathcal{Y}, \mathcal{Z}) \cdot \text{vec}(\mathcal{G}_i(\varepsilon, \mathcal{Y})) \right\}, \\ = & \min_{K^P \in \bar{K}^P} \max_{K^E \in \bar{K}^E} \left\{ x_0^T \left(\sum_{i=1}^k \mu_i \frac{d}{d\varepsilon} \mathcal{E}_i(\varepsilon) \right) x_0 + \sum_{i=1}^k \mu_i \frac{d}{d\varepsilon} \mathcal{T}_i(\varepsilon) \right. \\ & \left. + x_0^T \left(\sum_{i=1}^k \mu_i \mathcal{F}_i(\varepsilon, \mathcal{Y}, K^P, K^E) \right) x_0 + \sum_{i=1}^k \mu_i \mathcal{G}_i(\varepsilon, \mathcal{Y}) \right\}. \quad (39)\end{aligned}$$

It is important to observe that

$$\begin{aligned}& \sum_{i=1}^k \mu_i \mathcal{F}_i(\varepsilon, \mathcal{Y}, K^P, K^E) \\ = & - [A(\varepsilon) + B^P(\varepsilon)K^P + B^E(\varepsilon)K^E]^T \sum_{i=1}^k \mu_i \mathcal{Y}_i \\ & - \sum_{i=1}^k \mu_i \mathcal{Y}_i [A(\varepsilon) + B^P(\varepsilon)K^P + B^E(\varepsilon)K^E] \\ & - \mu_1 Q(\varepsilon) - \mu_1 K^{PT} R^P(\varepsilon) K^P + \mu_1 K^{ET} R^E(\varepsilon) K^E \\ & - \sum_{i=2}^k \mu_i \sum_{j=1}^{i-1} \frac{2i!}{j!(i-j)!} \mathcal{Y}_j G(\varepsilon) W G^T(\varepsilon) \mathcal{Y}_{i-j}, \\ & \sum_{i=1}^k \mu_i \mathcal{G}_i(\varepsilon, \mathcal{Y}) = - \sum_{i=1}^k \mu_i \text{Tr} \{ \mathcal{Y}_i G(\varepsilon) W G^T(\varepsilon) \}.\end{aligned}$$

Differentiating the expression within the bracket of (39) with respect to K^P and K^E yield the necessary conditions for an extremum of the performance index (34) on $[t_0, \varepsilon]$,

$$\begin{aligned}-2B^{PT}(\varepsilon) \sum_{i=1}^k \mu_i \mathcal{Y}_i M_0 - 2\mu_1 R^P(\varepsilon) K^P M_0 &= 0, \\ -2B^{ET}(\varepsilon) \sum_{i=1}^k \mu_i \mathcal{Y}_i M_0 + 2\mu_1 R^E(\varepsilon) K^E M_0 &= 0.\end{aligned}$$

Because M_0 is an arbitrary rank-one matrix, it must be true

$$K^P(\varepsilon, \mathcal{Y}, \mathcal{Z}) = -(R^P)^{-1}(\varepsilon) B^{PT}(\varepsilon) \sum_{r=1}^k \hat{\mu}_r \mathcal{Y}_r, \quad (40)$$

$$K^E(\varepsilon, \mathcal{Y}, \mathcal{Z}) = (R^E)^{-1}(\varepsilon) B^{ET}(\varepsilon) \sum_{r=1}^k \hat{\mu}_r \mathcal{Y}_r, \quad (41)$$

where $\hat{\mu}_r = \mu_i/\mu_1$ for $\mu_1 > 0$. Substituting the gain expressions (40) and (41) into the right member of the HJJ equation (39) yields the value of the minimum

$$\begin{aligned}
& x_0^T \left[\sum_{i=1}^k \mu_i \frac{d}{d\varepsilon} \mathcal{E}_i(\varepsilon) - A^T(\varepsilon) \sum_{i=1}^k \mu_i \mathcal{Y}_i - \sum_{i=1}^k \mu_i \mathcal{Y}_i A(\varepsilon) \right. \\
& - \mu_1 Q(\varepsilon) + \sum_{r=1}^k \hat{\mu}_r \mathcal{Y}_r B^P(\varepsilon) (R^P)^{-1}(\varepsilon) B^{PT}(\varepsilon) \sum_{i=1}^k \mu_i \mathcal{Y}_i \\
& + \sum_{i=1}^k \mu_i \mathcal{Y}_i B^P(\varepsilon) (R^P)^{-1}(\varepsilon) B^{PT}(\varepsilon) \sum_{s=1}^k \hat{\mu}_s \mathcal{Y}_s \\
& - \sum_{r=1}^k \hat{\mu}_r \mathcal{Y}_r B^E(\varepsilon) (R^E)^{-1}(\varepsilon) B^{ET}(\varepsilon) \sum_{i=1}^k \mu_i \mathcal{Y}_i \\
& - \sum_{i=1}^k \mu_i \mathcal{Y}_i B^E(\varepsilon) (R^E)^{-1}(\varepsilon) B^{ET}(\varepsilon) \sum_{s=1}^k \hat{\mu}_s \mathcal{Y}_s \\
& - \mu_1 \sum_{r=1}^k \hat{\mu}_r \mathcal{Y}_r B^P(\varepsilon) (R^P)^{-1}(\varepsilon) B^{PT}(\varepsilon) \sum_{s=1}^k \hat{\mu}_s \mathcal{Y}_s \\
& + \mu_1 \sum_{r=1}^k \hat{\mu}_r \mathcal{Y}_r B^E(\varepsilon) (R^E)^{-1}(\varepsilon) B^{ET}(\varepsilon) \sum_{s=1}^k \hat{\mu}_s \mathcal{Y}_s \\
& \left. - \sum_{i=2}^k \mu_i \sum_{j=1}^{i-1} \frac{2i!}{j!(i-j)!} \mathcal{Y}_j G(\varepsilon) W G^T(\varepsilon) \mathcal{Y}_{i-j} \right] x_0 \\
& + \sum_{i=1}^k \mu_i \frac{d}{d\varepsilon} \mathcal{T}_i(\varepsilon) - \sum_{i=1}^k \mu_i \text{Tr} \{ \mathcal{Y}_i G(\varepsilon) W G^T(\varepsilon) \} . \quad (42)
\end{aligned}$$

It is now necessary to exhibit time dependent functions $\{\mathcal{E}_i(\cdot)\}_{i=1}^k$ and $\{\mathcal{T}_i(\cdot)\}_{i=1}^k$ which will render the left side of (42) equal to zero for $\varepsilon \in [t_0, t_f]$, when $\{\mathcal{Y}_i\}_{i=1}^k$ are evaluated along solution trajectories of the cumulant-generating equations. Studying the expression (42) reveals that $\mathcal{E}_i(\cdot)$ and $\mathcal{T}_i(\cdot)$ for $1 \leq i \leq k$ satisfying the backward-in-time differential equations

$$\begin{aligned}
& \frac{d}{d\varepsilon} \mathcal{E}_1(\varepsilon) = A^T(\varepsilon) \mathcal{H}_1(\varepsilon) + \mathcal{H}_1(\varepsilon) A(\varepsilon) + Q(\varepsilon) \\
& - \mathcal{H}_1(\varepsilon) B^P(\varepsilon) (R^P)^{-1}(\varepsilon) B^{PT}(\varepsilon) \sum_{s=1}^k \hat{\mu}_s \mathcal{H}_s(\varepsilon) \\
& - \sum_{r=1}^k \hat{\mu}_r \mathcal{H}_r(\varepsilon) B^P(\varepsilon) (R^P)^{-1}(\varepsilon) B^{PT}(\varepsilon) \mathcal{H}_1(\varepsilon) \\
& + \mathcal{H}_1(\varepsilon) B^E(\varepsilon) (R^E)^{-1}(\varepsilon) B^{ET}(\varepsilon) \sum_{s=1}^k \hat{\mu}_s \mathcal{H}_s(\varepsilon) \\
& + \sum_{r=1}^k \hat{\mu}_r \mathcal{H}_r(\varepsilon) B^E(\varepsilon) (R^E)^{-1}(\varepsilon) B^{ET}(\varepsilon) \mathcal{H}_1(\varepsilon) \\
& + \sum_{r=1}^k \hat{\mu}_r \mathcal{H}_r(\varepsilon) B^P(\varepsilon) (R^P)^{-1}(\varepsilon) B^{PT}(\varepsilon) \sum_{s=1}^k \hat{\mu}_s \mathcal{H}_s(\varepsilon) \\
& - \sum_{r=1}^k \hat{\mu}_r \mathcal{H}_r(\varepsilon) B^E(\varepsilon) (R^E)^{-1}(\varepsilon) B^{ET}(\varepsilon) \sum_{s=1}^k \hat{\mu}_s \mathcal{H}_s(\varepsilon) , \quad (43)
\end{aligned}$$

and, for $2 \leq i \leq k$

$$\begin{aligned}
& \frac{d}{d\varepsilon} \mathcal{E}_i(\varepsilon) = A^T(\varepsilon) \mathcal{H}_i(\varepsilon) + \mathcal{H}_i(\varepsilon) A(\varepsilon) \\
& - \mathcal{H}_i(\varepsilon) B^P(\varepsilon) (R^P)^{-1}(\varepsilon) B^{PT}(\varepsilon) \sum_{s=1}^k \hat{\mu}_s \mathcal{H}_s(\varepsilon) \\
& - \sum_{r=1}^k \hat{\mu}_r \mathcal{H}_r(\varepsilon) B^P(\varepsilon) (R^P)^{-1}(\varepsilon) B^{PT}(\varepsilon) \mathcal{H}_i(\varepsilon) \\
& + \mathcal{H}_i(\varepsilon) B^E(\varepsilon) (R^E)^{-1}(\varepsilon) B^{ET}(\varepsilon) \sum_{s=1}^k \hat{\mu}_s \mathcal{H}_s(\varepsilon) \\
& + \sum_{r=1}^k \hat{\mu}_r \mathcal{H}_r(\varepsilon) B^E(\varepsilon) (R^E)^{-1}(\varepsilon) B^{ET}(\varepsilon) \mathcal{H}_i(\varepsilon) \\
& + \sum_{j=1}^{i-1} \frac{2i!}{j!(i-j)!} \mathcal{H}_j(\varepsilon) G(\varepsilon) W G^T(\varepsilon) \mathcal{H}_{i-j}(\varepsilon) , \quad (44)
\end{aligned}$$

$$\frac{d}{d\varepsilon} \mathcal{T}_i(\varepsilon) = \text{Tr} \{ \mathcal{H}_i(\varepsilon) G(\varepsilon) W G^T(\varepsilon) \} , \quad 1 \leq i \leq k , \quad (45)$$

will work. Furthermore, at the boundary condition, it is necessary to have $\mathcal{W}(t_0, \mathcal{H}_0, \mathcal{D}_0) = \phi_0(t_0, \mathcal{H}_0, \mathcal{D}_0)$, or equivalently

$$\begin{aligned}
& x_0^T \sum_{i=1}^k \mu_i (\mathcal{H}_{i0} + \mathcal{E}_i(t_0)) x_0 + \sum_{i=1}^k \mu_i (\mathcal{D}_{i0} + \mathcal{T}_i(t_0)) \\
& = x_0^T \sum_{i=1}^k \mu_i \mathcal{H}_{i0} x_0 + \sum_{i=1}^k \mu_i \mathcal{D}_{i0} .
\end{aligned}$$

Thus, matching the boundary condition yields the corresponding initial value conditions $\mathcal{E}_i(t_0) = 0$ and $\mathcal{T}_i(t_0) = 0$ for the equations (43)-(45). Applying the feedback gains specified in (40) and (41) along the solution trajectories of the equations (32)-(33), these equations become Riccati-type equations

$$\begin{aligned}
& \frac{d}{d\varepsilon} \mathcal{H}_1(\varepsilon) = -A^T(\varepsilon) \mathcal{H}_1(\varepsilon) - \mathcal{H}_1(\varepsilon) A(\varepsilon) - Q(\varepsilon) \\
& + \mathcal{H}_1(\varepsilon) B^P(\varepsilon) (R^P)^{-1}(\varepsilon) B^{PT}(\varepsilon) \sum_{s=1}^k \hat{\mu}_s \mathcal{H}_s(\varepsilon) \\
& + \sum_{r=1}^k \hat{\mu}_r \mathcal{H}_r(\varepsilon) B^P(\varepsilon) (R^P)^{-1}(\varepsilon) B^{PT}(\varepsilon) \mathcal{H}_1(\varepsilon) \\
& - \mathcal{H}_1(\varepsilon) B^E(\varepsilon) (R^E)^{-1}(\varepsilon) B^{ET}(\varepsilon) \sum_{s=1}^k \hat{\mu}_s \mathcal{H}_s(\varepsilon) \\
& - \sum_{r=1}^k \hat{\mu}_r \mathcal{H}_r(\varepsilon) B^E(\varepsilon) (R^E)^{-1}(\varepsilon) B^{ET}(\varepsilon) \mathcal{H}_1(\varepsilon) \\
& - \sum_{r=1}^k \hat{\mu}_r \mathcal{H}_r(\varepsilon) B^P(\varepsilon) (R^P)^{-1}(\varepsilon) B^{PT}(\varepsilon) \sum_{s=1}^k \hat{\mu}_s \mathcal{H}_s(\varepsilon) \\
& + \sum_{r=1}^k \hat{\mu}_r \mathcal{H}_r(\varepsilon) B^E(\varepsilon) (R^E)^{-1}(\varepsilon) B^{ET}(\varepsilon) \sum_{s=1}^k \hat{\mu}_s \mathcal{H}_s(\varepsilon) , \quad (46)
\end{aligned}$$

and, for $2 \leq i \leq k$

$$\begin{aligned} \frac{d}{d\varepsilon} \mathcal{H}_i(\varepsilon) &= -A^T(\varepsilon) \mathcal{H}_i(\varepsilon) - \mathcal{H}_i(\varepsilon) A(\varepsilon) \\ &+ \mathcal{H}_i(\varepsilon) B^P(\varepsilon) (R^P)^{-1}(\varepsilon) B^{PT}(\varepsilon) \sum_{s=1}^k \hat{\mu}_s \mathcal{H}_s(\varepsilon) \\ &+ \sum_{r=1}^k \hat{\mu}_r \mathcal{H}_r(\varepsilon) B^P(\varepsilon) (R^P)^{-1}(\varepsilon) B^{PT}(\varepsilon) \mathcal{H}_i(\varepsilon) \\ &- \mathcal{H}_i(\varepsilon) B^E(\varepsilon) (R^E)^{-1}(\varepsilon) B^{ET}(\varepsilon) \sum_{s=1}^k \hat{\mu}_s \mathcal{H}_s(\varepsilon) \\ &- \sum_{r=1}^k \hat{\mu}_r \mathcal{H}_r(\varepsilon) B^E(\varepsilon) (R^E)^{-1}(\varepsilon) B^{ET}(\varepsilon) \mathcal{H}_i(\varepsilon) \\ &- \sum_{j=1}^{i-1} \frac{2i!}{j!(i-j)!} \mathcal{H}_j(\varepsilon) G(\varepsilon) W G^T(\varepsilon) \mathcal{H}_{i-j}(\varepsilon), \quad (47) \end{aligned}$$

together, for $1 \leq i \leq k$

$$\frac{d}{d\varepsilon} \mathcal{D}_i(\varepsilon) = -\text{Tr} \{ \mathcal{H}_i(\varepsilon) G(\varepsilon) W G^T(\varepsilon) \} \quad (48)$$

where the terminal conditions $\mathcal{H}_1(t_f) = Q_f$, $\mathcal{H}_i(t_f) = 0$ for $2 \leq i \leq k$ and $\mathcal{D}_i(t_f) = 0$ for $1 \leq i \leq k$. Thus, whenever these equations (46)-(48) admit solutions $\{\mathcal{H}_i(\cdot)\}_{i=1}^k$ and $\{\mathcal{D}_i(\cdot)\}_{i=1}^k$, then the existence of $\{\mathcal{E}_i(\cdot)\}_{i=1}^k$ and $\{\mathcal{T}_i(\cdot)\}_{i=1}^k$ satisfying the equations (43)-(45) are assured. By comparing equations (43)-(45) to those of (46)-(48), one may recognize that these sets of equations are related to one another by

$$\frac{d}{d\varepsilon} \mathcal{E}_i(\varepsilon) = -\frac{d}{d\varepsilon} \mathcal{H}_i(\varepsilon) \text{ and } \frac{d}{d\varepsilon} \mathcal{T}_i(\varepsilon) = -\frac{d}{d\varepsilon} \mathcal{D}_i(\varepsilon)$$

for $1 \leq i \leq k$. Enforcing the initial value conditions of $\mathcal{E}_i(t_0) = 0$ and $\mathcal{T}_i(t_0) = 0$ uniquely implies that

$$\mathcal{E}_i(\varepsilon) = \mathcal{H}_i(t_0) - \mathcal{H}_i(\varepsilon) \text{ and } \mathcal{T}_i(\varepsilon) = \mathcal{D}_i(t_0) - \mathcal{D}_i(\varepsilon)$$

for all $\varepsilon \in [t_0, t_f]$ and yields a value function

$$\begin{aligned} \mathcal{W}(\varepsilon, \mathcal{Y}, \mathcal{Z}) &= \mathcal{V}(\varepsilon, \mathcal{Y}, \mathcal{Z}) \\ &= x_0^T \sum_{i=1}^k \mu_i \mathcal{H}_i(t_0) x_0 + \sum_{i=1}^k \mu_i \mathcal{D}_i(t_0), \end{aligned}$$

for which the sufficient condition (36) of the verification theorem is satisfied. Therefore, the feedback gains for pursuit team P , (40) and evader team E , (41) optimizing the performance index stated in (34) become optimal

$$K^{P*}(\varepsilon) = -(R^P)^{-1}(\varepsilon) B^{PT}(\varepsilon) \sum_{r=1}^k \hat{\mu}_r \mathcal{H}_r^*(\varepsilon), \quad (49)$$

$$K^{E*}(\varepsilon) = (R^E)^{-1}(\varepsilon) B^{ET}(\varepsilon) \sum_{r=1}^k \hat{\mu}_r \mathcal{H}_r^*(\varepsilon). \quad (50)$$

Theorem 7: Strategies for Multi-Player Pursuit-Evasion. Consider the multi-player pursuit-evasion game as described by (14)-(15) where the pairs (A, B^P) and (A, B^E) are uniformly stabilizable. Fix $\xi^P \in W^P$, $\xi^E \in W^E$, $k \in \mathbb{Z}^+$,

and $\mu = \{\mu_i \geq 0\}_{i=1}^k$ with $\mu_1 > 0$. Then the saddle-point solution is achieved by the non-inferior strategy gains

$$K^{P*}(\alpha) = -(R^P)^{-1}(\alpha) B^{PT}(\alpha) \sum_{r=1}^k \hat{\mu}_r \mathcal{H}_r^*(\alpha), \quad (51)$$

$$K^{E*}(\alpha) = (R^E)^{-1}(\alpha) B^{ET}(\alpha) \sum_{r=1}^k \hat{\mu}_r \mathcal{H}_r^*(\alpha), \quad (52)$$

where additional parametric design freedom $\hat{\mu}_r$ mutually chosen by rival teams represent different levels of influence as they deem important to the global cost distribution and $\{\mathcal{H}_r^*(\alpha) \geq 0\}_{r=1}^k$ solve the coupled differential equations

$$\begin{aligned} \frac{d}{d\alpha} \mathcal{H}_1^*(\alpha) &= \\ &- [A(\alpha) + B^P(\alpha) K^{P*}(\alpha) + B^E(\alpha) K^{E*}(\alpha)]^T \mathcal{H}_1^*(\alpha) \\ &- \mathcal{H}_1^*(\alpha) [A(\alpha) + B^P(\alpha) K^{P*}(\alpha) + B^E(\alpha) K^{E*}(\alpha)] \\ &\quad - Q(\alpha) - K^{P*T}(\alpha) R^P(\alpha) K^{P*}(\alpha) \\ &\quad + K^{E*T}(\alpha) R^E(\alpha) K^{E*}(\alpha), \quad \mathcal{H}_1^*(t_f) = Q_f \quad (53) \end{aligned}$$

and, for $2 \leq r \leq k$ with $\mathcal{H}_r^*(t_f) = 0$

$$\begin{aligned} \frac{d}{d\alpha} \mathcal{H}_r^*(\alpha) &= \\ &- [A(\alpha) + B^P(\alpha) K^{P*}(\alpha) + B^E(\alpha) K^{E*}(\alpha)]^T \mathcal{H}_r^*(\alpha) \\ &- \mathcal{H}_r^*(\alpha) [A(\alpha) + B^P(\alpha) K^{P*}(\alpha) + B^E(\alpha) K^{E*}(\alpha)] \\ &- \sum_{s=1}^{r-1} \frac{2r!}{s!(r-s)!} \mathcal{H}_s^*(\alpha) G(\alpha) W G^T(\alpha) \mathcal{H}_{r-s}^*(\alpha), \quad (54) \end{aligned}$$

V. CONCLUSIONS

This paper dealt with a multi-player pursuit-evasion differential game modeled in a stochastic environment for realistic conditions. Matrix differential equations for generating statistics of the standard integral-quadratic performance-measure used in this game were derived. A direct dynamic programming approach was used to solve for saddle-point solutions that can address both control strategy selection and performance analysis aspects. Hopefully, these results will make some new theoretical contributions and performance analysis tools to stochastic differential game communities.

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